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On the Leray–Maslov quantization of Lagrangian submanifolds

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We define a generalized Maslov index on $\Lambda_4(n)$, the covering of order 4 of the Lagrangian Grassmannian $\Lambda(n)$. That generalized Maslov index, defined on all pairs of elements of $\Lambda_4(n)$, allows us to state Leray–Maslov's quantization rule independently of the choice of frame.

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1. Introduction

In his treatise "Lagrangian Analysis and Quantum Mechanics" [4] Leray gives a quantization condition for Lagrangian submanifolds V of $Z=\mathbb{R}_q^n\times\mathbb{R}_p^n$, equipped with the symplectic form $\omega=\mathrm{d}q\wedge\mathrm{d}p$. That condition [4, ch. II, §3.6, definition 6.2] can be stated as follows:

(1) For every two-frame R the function

$$\check{V}\backslash\check{\Sigma}_R\ni\check{z}\to\tfrac{1}{4}m_R(\check{z})+\frac{\nu_0}{2\pi\mathrm{i}}\varphi_R(\check{z})\in R/4\mathbb{Z} \tag{1.1}$$

is defined modulo 1 on $V\backslash\Sigma_R$,

where $\check{V}$ is the universal covering manifold of V, $\check{\Sigma}_R$ the apparent contour of $\check{V}$ relative to the frame R, Σ_R its projection, φ_R the phase of RV and m_R is the Maslov index on V relative to R and i/ν_0 is Planck's constant (see ref. [4, ch. I, §§2.5, 3.2, 3.3] for these notions).

Condition (1) is independent of the choice of the frame R: if it holds in one frame it holds in every frame; however, *its statement requires a choice of frame.*

We show in this paper that it is possible to give a quantization condition [condition (2) below] which is equivalent to (1), but which does not require a pre-

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liminary choice of frame. This is made possible by the use of a Maslov index, free of any transversality assumption, and which is a slight variant of the Maslov index we defined in our papers [2,3]. Condition (2) thus obtained can be written

$$\tfrac{1}{4}[m_\gamma]+\frac{\nu_0}{2\pi \mathrm{i}}C_\gamma\in\mathbb{Z}_4\,,\quad \gamma\in\pi_1(V)\,, \tag{1.2}$$

where $C_\gamma=\oint_\gamma p\,\mathrm{d}q$, m_γ is the jump of the Maslov index along γ. That condition when applied to the classical trajectories of the harmonic oscillator very simply yields the energy levels predicted by quantum mechanics; it has in fact a great similarity with the classical Bohr–Sommerfeld quantization condition; we therefore call condition (2) the "generalized Bohr–Sommerfeld quantization condition".

Our notations are essentially those of ref. [3]. For $x\in\mathbb{R}$ we denote by $\hat{x}$ ($\dot{x}$) the class of x modulo 4 (modulo 8). Let $Z=\mathbb{R}^n\times\mathbb{R}^n$ be equipped with the symplectic form $\omega(z,z')=x'\cdot y-x\cdot y'$, $z=(x,y)$, $z'=(x',y')$. We denote by Λ the Lagrangian Grassmannian of (Z,ω): $l\in\Lambda$ if and only if l is a n-dimensional subspace of Z, and $\omega=0$ on l. Sp is the symplectic group of (Z,ω), i.e., the group of all automorphisms of Z leaving ω invariant; Mp is the metaplectic group, i.e., the unitary representation in $L^2(\mathbb{R}^n)$ of Sp_2, the double cover of Sp. Mp is generated by generalized Fourier transforms S_A given by

$$S_Af(x)=\left(\frac{|\nu|}{2\pi\mathrm{i}}\right)^{n/2}\Delta(A)\int\limits_{\mathbb{R}^n}\mathrm{e}^{\nu A(x,x')}f(x)\,\mathrm{d}x\,,$$

with $A(x,x')=\frac{1}{2}Px\cdot x-Lx\cdot x'+\frac{1}{2}Qx'\cdot x'$, $P=P^{\mathrm{T}}$, $Q=Q^{\mathrm{T}}$, $\det(L)\neq0$, $\Delta(A)=\mathrm{i}^m|\det(L)|^{1/2}$, $m\pi=\operatorname{Arg}(\det L)\bmod 2\pi$ (see ref. [4, ch. I]), $\nu\in\mathrm{i}[1,\infty)$ being a parameter. Finally, for $q=2,3,\ldots$ we write $\mathbb{Z}_q=\mathbb{Z}/q\mathbb{Z}$.

2. Extended Arnold–Leray–Maslov indices

We denote as in ref. [3] the signature of a triple (l,l',l'') of Lagrangian planes by $\sigma(l,l',l'')$ and its class modulo 8 by $\dot{\sigma}(l,l',l'')$; recall [1–3] that σ is a Sp-invariant, antisymmetric, $\mathbb{Z}$-valued cocycle on $(\Lambda(n))^3$, locally constant on each set

$$\Lambda_{k,k',k''}=\{(l,l',l''):\dim(l\cap l')=k,\ \dim(l'\cap l'')=k',\ \dim(l\cap l'')=k''\}$$
$$\text{for } 0\leqslant k,k',k''\leqslant n\,.$$

Furthermore

$$\sigma(l,l',l'')\equiv n+\dim(l\cap l')+\dim(l'\cap l'')+\dim(l\cap l'')\bmod 2\,. \tag{2.1}$$

As Dazord [5] we define the modified signature d by

$$d(l,l',l'')=\tfrac{1}{2}(-n-\dim(l\cap l')-\dim(l'\cap l'')+\dim(l\cap l'')$$
$$+\sigma(l,l',l'')) . \qquad (2.2)$$

It is immediately clear by (2.1) and the properties of σ that

Proposition 2.1. The Dazord signature d is a Sp-invariant $\mathbb{Z}$-valued cocycle, locally constant on each $\Lambda_{k,k',k''}$.

Observe that due to the presence of the minus sign before $\dim(l\cap l'')$, d no longer is antisymmetric; however, $d(l,l',l'')=d(l'',l',l)$. For fixed $l\in\Lambda$ we define

$$\hat{d}_l(S,S')=\hat{d}_l(s,s')=\hat{d}(l,sl,ss'l)\in\mathbb{Z}_4 , \qquad (2.3)$$

s, $s^1\in\mathrm{Sp}$ being the projections of S, S'; it is clear from proposition 2.1 that

$$\hat{d}_l(SS',S'')+\hat{d}_l(S,S')=\hat{d}_l(S,S',S'')+\hat{d}_l(S',S'') , \qquad (2.4)$$

i.e., d_l is a $\mathbb{Z}_4$-valued cocycle on the group Mp. Define now

$$m_l(S)=\tfrac{1}{2}(\dot{\mu}_l(S)+\dot{n}+\dot{\dim}(sl\cap l))\in\mathbb{R}/4\mathbb{Z} , \qquad (2.5)$$

where $\dot{\mu}_l$ is the Leray–Maslov index on Mp relative to l [3, §3, p. 268]; it immediately follows from the properties of $\dot{\mu}_l$ [3, thm. 3.2] that $m_l(S)\in\mathbb{Z}_4$ and that

Theorem 2.2.

(a) The function m_l is the only function $Mp\mapsto\mathbb{Z}_4$ having the folowing properties:

(1) $m_l(SS')-m_l(S)-m_l(S')=\hat{d}_l(S,S')$ (i.e., d_l is a coboundary of m_l);

(2) the mapping $(S,l')\to m_l(S)-\hat{d}(sl,l,l')\in\mathbb{Z}_4$ is locally constant for $sl\cap l'=l\cap l'=0$, hence m_l is locally constant for $sl\cap l=\{0\}$.

(b) Furthermore m_l has the following properties:

(3) $m_l(S)+m_l(S^{-1})=\hat{n}+\widehat{\dim}(l\cap sl)$, $m_l(I)=\hat{n}$,

(4) $m_l(-S)=m_l(S)+\hat{2}$,

(5) $m_l(S)-m_{l'}(S)=\hat{d}(sl,l,l')-\hat{d}(sl,sl',l')+\widehat{\dim}(sl\cap l)-\widehat{\dim}(sl'\cap l')$.

It is instructive to compute $m_l(S)$ when $l=\{0\}\times\mathbb{R}^n$ and $S=S_A$; then $\dot{\mu}_l(S_A)=\dot{\mu}_0(S_A)=2\dot{m}(A)-\dot{n}$ [3, formula (3.12) thm. 3.2(i)]; thus

$$m_0(S_A)=m_{l_0}(S_A)=m(A) , \qquad (2.6)$$

hence $m_{l_0}=m_0$ is the Arnold–Maslov index modulo 4 [4, ch. I, §2,8]. This motivates following definition:

Definition 2.3. We call $m_l:\mathrm{Mp}\to\mathbb{Z}_4$ the extended Arnold–Leray–Maslov (ALM) index on Mp, relative to l.

Let now $\hat{l}$, $\hat{l}'$ be two elements of Λ_4, the covering space of order 4 of $\Lambda=\Lambda(n)$; there exists $S\in\mathrm{Mp}$ such that $\hat{l}'=S\hat{l}$ and if $S'\in\mathrm{Mp}$ is such that $\hat{l}'=S'\hat{l}$, then $S'=SH$,

H being in the stabilizer $\mathrm{St}(l)$ of l in Mp; from theorem 2.2(1) and the definition (2.3) of $\hat{d}_l$ it follows that

$$m_l(S') = m_l(S) + m_l(H) + \hat{d}(l, sl, sl) ;$$

hence, using (2.2) and the antisymmetry of σ,

$$m_l(S') = m_l(S) + m_l(H) = m_l(S) + m_l(H) .$$

Now, $m_l(H) = \frac{1}{2}\dot{\mu}_l(H) = \hat{0}$ in view of lemma 4.1 of ref. [3]; hence $m_l(S)$ is independent of the choice of S in Mp such that $\hat{l}' = S\hat{l}$ and only depends on the pair $(\hat{l}, \hat{l}')$; we therefore denote it $m(\hat{l}, \hat{l}')$; the properties of $m(\hat{l}, \hat{l}')$ follow from theorem 2.2 (compare with ref. [3, thms. 4.1 and 4.2]):

Theorem 2.4.

(a) The function

$$m : \Lambda_4^2 \ni (\hat{l}, \hat{l}') \mapsto m(\hat{l}, \hat{l}') \in \mathbb{Z}_4$$

is the only function $\Lambda_4^2 \to \mathbb{Z}_4$ which is locally constant on $\{(\hat{l}, \hat{l}') : l \cap l' = \{0\}\}$ and such that

(1) $m(\hat{l}, \hat{l}') - m(\hat{l}, \hat{l}'') + m(\hat{l}', \hat{l}'') = \hat{n} + \hat{d}(l'', l', l)$.

(b) m has furthermore the following properties:

(2) $m(\hat{l}, \hat{l}') + m(\hat{l}', \hat{l}) = \hat{n} - \widehat{\dim}(l \cap l')$, $m(\hat{l}, \hat{l}) = \hat{0}$,

(3) $m(S\hat{l}, S\hat{l}') = m(\hat{l}, \hat{l}')$, $\forall S \in Mp$.

Using the same argument as in ref. [3, thm. (4.2)(iii)], one readily proves that the following identity holds, where $\mathrm{i} = \sqrt{-1}$, $k \in \mathbb{Z}$:

$$m(i^k\hat{l}, i^{k'}\hat{l}') = m(\hat{l}, \hat{l}') + \hat{k} - \hat{k}' ; \tag{2.7}$$

note that (2.7) then appears as a particular case of property (3) of theorem 2.4 when $k = k'$.

For a triple of pairwise transverse lagrangian planes (l, l', l''), one defines [4, ch. I, §2,4] the index of inertia $i(l, l', l'')$ as being the index of inertia of the quadratic form on l (or l', or l'') defined by $Q(z) = \omega(z, z')$ (or $Q'(z') = \omega(z', z'')$, or $Q''(z'') = \omega(z'', z)$) for $(z, z', z'') \in l \times l' \times l''$, $z + z' + z'' = 0$. That index is related to the signature $\sigma(l, l', l'')$ by

$$\sigma(l, l', l'') = 2\mathrm{i}(l, l', l'') - n \quad \text{if } l \cap l' = l' \cap l'' = l \cap l'' = \{0\} ;$$

from that relation and theorem 2.4(a) it follows that

Corollary 2.5. *The restriction m' of m to $(\Lambda_4^2)' = \{(\hat{l}, \hat{l}') : l \cap l' = \{0\}\}$ is the only locally constant function $(\Lambda_4^2)' \to \mathbb{Z}_4$ such that:*

$$m'(\hat{l}, \hat{l}') - m'(\hat{l}, \hat{l}'') + m'(\hat{l}', \hat{l}'') = i(l, l', l'')$$

for $l \cap l' = l' \cap l'' = l'' \cap l = \{0\}$.

Proof. Immediate in view of theorem 2.4(a) since we have

$$\begin{aligned}\hat{n} + \hat{d}(l'', l', l) &= n + \tfrac{1}{2}(-\hat{n} + \dot{\sigma}(l'', l', l)) \\ &= \hat{i}(l, l', l''),\end{aligned}$$

when $l \cap l' = l' \cap l'' = l'' \cap l = \{0\}$. □

Corollary 2.5 *identifies* m' with the index on Λ_4 defined by Leray [4, ch. I, §2,5, §2,8, thm. 8]; Leray's index was an extension of Maslov's index, whose correct definition had been given by Arnold [6], thus justifying

Definition 2.6. We call $m : \Lambda_4^2 \to \mathbb{Z}_4$ the extended ALM index on Λ_4.

For fixed $l_0 \in \Lambda$ consider the multiplication on $\mathrm{Sp} \times \mathbb{Z}_4$ defined by

$$(s, \hat{m}) \cdot (s', \hat{m}') = (ss', \hat{m} + \hat{m}' + \hat{d}_{l_0}(s, s')) ; \tag{2.8}$$

in view of the cocycle property of $\hat{d}_{l_0}$ it is clear that $\mathrm{Sp} \times \mathbb{Z}_4$ is a group G_{l_0} for that multiplication; $(I, \hat{n})$ is the identity of that group and $(s, \hat{m})^{-1} = (s^{-1}, \hat{n} - \hat{m} + \widehat{\dim}(sl \cap l))$.

Let $l \in \Lambda$, $S \in \mathrm{Mp}$; in view of theorem 2.2(4) the image of $m_l(S)$ in $\mathbb{Z}_2$ only depends on l and on the projection s of S; we denote that image $m_l\langle s \rangle$.

Theorem 2.7. *For every* $l_0 \in \Lambda$ *the mapping*

$$Mp \ni S \mapsto (s, m_{l_0}(S)) \in G_{l_0}$$

is an isomorphism of Mp onto the subgroup $\langle \mathrm{Sp} \times \Lambda_4 \rangle_{l_0} = \{(s, \hat{m}); m \in m_{l_0}\langle s \rangle\}$ *of* G_l; *the restriction of that mapping to* $\{S : sl_0 \cap l_0 = \{0\}\}$ *is a homeomorphism.*

Proof. Absolutely similar to the proof of theorem 5.1 in ref. [3]. □

We may thus, for given l, identify Mp with $\langle \mathrm{Sp} \times \Lambda_4 \rangle_{l_0}$, that identification being both algebraic and topological (see ref. [3, thm. 5.2]). Similarly:

Theorem 2.8. *For every* $l_0 \in \Lambda$ *the mapping*

$$\Lambda_4 \in \hat{l} \mapsto (l, m(\hat{l}, \hat{l}_0)) \in \Lambda \times \mathbb{Z}_4 \tag{$*$}$$

is a bijection. The restriction of that bijection to $\Lambda_{(0)} = \{\hat{l} : l \cap l_0 = \{0\}\}$ *is a homeomorphism.*

Proof. The mapping ($*$) is injective for, if $l=l'$, then $\hat{l}=\mathrm{i}^k\hat{l}'$ for some $k\in\mathbb{Z}$, and $m(\hat{l},\hat{l}_0)=m(\hat{l}',\hat{l}_0)=m(\hat{l},\hat{l}_0)+\hat{k}$ then implies $k\equiv 0$ mod 4, hence $\hat{l}=\hat{l}'$. The mapping ($*$) is obviously surjective. The second statement follows from the fact that m is locally constant on $\Lambda_{(0)}$ [theorem 2.4(a)] and hence continuous. □

We may thus identify Λ_4 with $\Lambda\times\mathbb{Z}_4$; transporting the topology of Λ_4 on that set the identification also becomes topological; we denote by $\langle\Lambda\times\mathbb{Z}_4\rangle_{l_0}$ the topological space thus defined.

Using the identification $\Lambda_4=\langle\Lambda\times\mathbb{Z}_4\rangle_{l_0}$ it is easy to calculate explicitly the ALM index:

Corollary 2.9. *For $\hat{l}=(l,\hat{\lambda})$, $\hat{l}'=(l',\hat{\lambda}')$ in $\langle\Lambda\times\mathbb{Z}_4\rangle_{l_0}$ we have:*
(1) $m(\hat{l},\hat{l}')=\hat{\lambda}-\hat{\lambda}'+\hat{n}+\hat{d}(l_0,l,l')$;
hence in particular
(2) $m(\hat{l},\hat{l}_0)=\hat{\lambda}-\hat{\lambda}_0+\widehat{\dim}(l\cap l_0)$ for $\hat{l}_0=(l_0,\hat{\lambda}_0)$.

Proof. Obvious in view of theorem 2.8 and formula (1) in theorem 2.4. □

Sp acts transitively and continuously on Λ; that action is covered by an action of Mp on Λ_4 (more generally the covering group Sp_q of order q acts on the covering space Λ_{2q} of order $2q$, see ref. [4, ch. I, §2,3, thm. 3, 3°]. The corresponding action of $\langle\mathrm{Sp}\times\mathbb{Z}_4\rangle_{l_0}$ on $\langle\Lambda\times\mathbb{Z}_4\rangle_{l_0}$ is easily described using the Maslov index.

Theorem 2.10. *The group $\langle\mathrm{Sp}\times\mathbb{Z}_4\rangle_{l_0}$ acts continuously and transitively on $\langle\Lambda\times\mathbb{Z}_4\rangle_{l_0}$ via*

$$(s,\hat{\mu})(l,\hat{\lambda})=(sl,\hat{\mu}+\hat{\lambda}+\hat{d}(l_0,sl_0,sl)) .$$

Proof. Obvious in view of the cocycle property of d, the continuity of the action being a consequence of proposition 2.1 and the definition of the topologies of $\langle\mathrm{Sp}\times\mathbb{Z}_4\rangle_{l_0}$, $\langle\Lambda\times\mathbb{Z}_4\rangle_{l_0}$.

3. A quantization condition for Lagrangian manifolds in Z

Let V be a *connected* lagrangian manifold in Z, $\check{V}$ its universal covering space. We assume that V ($\check{V}$) is two-oriented, i.e., that there exists a continuous mapping $V\ni z\mapsto\hat{l}(z)\in\Lambda_4$ ($\check{V}\ni\check{z}\mapsto\hat{l}(\check{z})\in\Lambda_4$) (a "two-orientation") which composed with the natural projection $\Lambda_4\to\Lambda$ gives the mapping $z\to l(z)=T_zV$ ($\check{z}\to l(\check{z})=T_{\check{z}}\check{V}$) (the tangent space to V ($\check{V}$) at z ($\check{z}$)).

For each $\hat{l}_0\in\Lambda_4$ we define $m_{\hat{l}_0}:\check{V}\to\mathbb{Z}_4$ by:

3.1. *$m_{\hat{l}_0}(\check{z}) = m(\hat{l}(\check{z}), \hat{l}_0)$ [$=\hat{\lambda}(\check{z}) - \hat{\lambda}_0 + \widehat{\dim}(l \cap l_0)$ if one identifies Λ_4 with $\langle \Lambda \times \mathbb{Z}_4 \rangle_{l_0}$, $\hat{l}(\check{z})$ with $(l(\check{z}), \hat{\lambda}(\check{z}))$, $\hat{l}_0$ with $(\hat{l}_0, \hat{\lambda}_0)$].*

The index $m_{\hat{l}_0}$ has the following properties:

$$m_{\hat{l}_0}(\check{z}) - m_{\hat{l}_1}(\check{z}) = \hat{n} + \hat{d}(l(z), l_0, l_1) - m(\hat{l}_0, \hat{l}_1) . \tag{3.1}$$

Proof. Formula (1) in theorem 2.4(a). □

3.2. *$m_{\hat{l}_0}(\cdot)$ is constant on each connected component of the set $\check{V} \backslash \check{\Sigma}_{l_0}$, where $\check{\Sigma}_{l_0} = \{z : l(z) \cap l_0 = \{0\}\}$ is the "apparent contour" of V relative to l_0.*

Proof. Obvious again by theorem 2.4(a), since the mapping $\check{z} \to \hat{l}(\check{z})$ is continuous. □

The first homotopy group $\pi_1(V)$ acts on V: if $\gamma \in \pi_1(V)$, $\check{z} \in \check{V}$, then $\gamma\check{z} \in \check{V}$ and has the same projection $z \in V$ as $\check{z}$. The following result is essential:

Proposition 3.3. *The difference $m_{\hat{l}_0}(\gamma\check{z}) - m_{\hat{l}_0}(\check{z})$ only depends on $\gamma \in \pi_1(V)$; it is therefore denoted $\hat{m}_\gamma$; clearly $\hat{m}_\gamma \in Z_4$.*

Proof. In view of formula (1) in theorem 2.4 we have

$$-m(\hat{l}(\gamma\check{z}), \hat{l}_0) + m(\hat{l}(\check{z}), \hat{l}_0) = \hat{n} + \hat{d}(l_0, l(\check{z}), l(\gamma\check{z})) - m(\hat{l}(\gamma\check{z}), \hat{l}(\check{z})) .$$

Now, both mappings $z \mapsto \hat{l}(\gamma\check{z})$ and $\check{z} \to \hat{l}(\check{z})$ are continuous, and cover the mapping $\check{z} \mapsto l(\check{z}) \in T_z\check{V}$, since $\check{z}$ and $\gamma\check{z}$ have the same projection $z \in V$; it follows that

$$-m(\hat{l}(\gamma\check{z}), \hat{l}_0) + m(\hat{l}(\check{z}), \hat{l}_0) = -m(\hat{l}(\gamma\check{z}), \hat{l}(\check{z})) ,$$

noting that $d(l_0, l(\check{z}), l(\gamma\check{z})) = -n$, σ being antisymmetric. The Lagrangian manifold $\check{V}$ is connected since V is, hence $(\hat{l}(\gamma\check{z}), \hat{l}(\check{z}))$ describes a connected subset of $\Lambda_4 \times \Lambda_4$; this implies that $m(\hat{l}(\gamma\check{z}), \hat{l}(\check{z}))$ has a constant value in view of theorem 2.4(a). □

The symplectic form $\omega = \mathrm{d}p \wedge \mathrm{d}q$ vanishes on every Lagrangian manifold V; hence there exists a function $\varphi : \check{V} \to \mathbb{R}$, the *phase* of V, such that $\mathrm{d}\varphi = \langle p, \mathrm{d}x \rangle$ (*sensu stricto*, φ is only determined up to an additive constant since V is connected).

Let $\nu_0 \in \mathrm{i}[1, \infty)$ be a purely imaginary number (in quantum mechanics ν_0 would be $2\pi\mathrm{i}/h$, h Planck's constant), and consider the function

$$F(\check{z}) = \tfrac{1}{4} m_{l_0}(\check{z}) + \frac{\nu_0}{2\pi\mathrm{i}} \hat{\varphi}(\check{z}) . \tag{3.2}$$

Replacing $\check{z}$ by $\gamma\check{z}$, $\gamma \in \pi_1(V)$, the phase $\varphi(\check{z})$ increases by a constant value, denoted C_γ and depending only on γ; in fact

$$C_\gamma = \varphi(\gamma\check{z}) - \varphi(\check{z}) = \oint_\gamma \langle p, \mathrm{d}q \rangle \ ; \tag{3.3}$$

hence, taking into account proposition 3.3,

$$F(\gamma\check{z}) - F(\check{z}) = \tfrac{1}{4}\hat{m}_\gamma + \frac{\nu_0}{2\pi \mathrm{i}} \hat{C}_\gamma \, . \tag{3.4}$$

Definition 3.4. The connected Lagrangian manifold V satisfies the generalized Bohr–Sommerfeld quantization condition if

$$\tfrac{1}{4}\hat{m}_\gamma + \frac{\nu_0}{2\pi \mathrm{i}} \hat{C}_\gamma \in \mathbb{Z}_4 \tag{3.5}$$

for every $\gamma \in \pi_1(V)$.

Example. Consider the one-dimensional harmonic oscillator whose Hamiltonian function is given by

$$H(q,p) = \frac{1}{2m}(p^2 + m^2\omega^2 q^2) \quad (m, \omega > 0) \ . \tag{3.6}$$

The classical trajectories in phase space are solutions of Hamilton's equations $q(t) = \partial H/\partial p$, $p(t) = -\partial H/\partial q$ and are thus the ellipses

$$V_{a,\alpha}\!: \ \ q(t) = a\cos(\omega t + \alpha) \ , \quad p(t) = -ma\omega \sin(\omega t + \alpha) \ , \quad a, \alpha \text{ real} \, .$$

The universal cover $\check{V}_{a,\alpha}$ of $V_{a,\alpha}$ is parametrized by

$$\check{V}_{a,\alpha}\!: \begin{cases} q(t) = a\cos(\omega t + \alpha) \ , \\ p(t) = -ma\omega \sin(\omega t + \alpha) \ , \quad (t \in \mathbb{R}) \\ u(t) = \omega t \ . \end{cases}$$

Setting $\check{z} = (z, u) = (q, p, u)$, the differential $\mathrm{d}\varphi = p \, \mathrm{d}q$ is given by

$$\mathrm{d}\varphi(\check{z}) = \tfrac{1}{2} ma^2\omega^2 (1 - \cos 2\omega t) \ ,$$

that is,

$$\varphi(\check{z}) = \tfrac{1}{2}(ma^2\omega^2 u + pq) \ .$$

Let $\gamma : [0, 2\pi/\omega] \ni t \mapsto (a\cos(\omega t + \alpha), \ -ma\omega \sin(\omega t + \alpha))$ be the generator of $\pi_1(V) = \pi_1(S^1) = (\mathbb{Z}, +)$; one immediately checks that $C_\gamma = ma^2\omega\pi$, and, using formula (1) in corollary 2.9 we get as well

$$\hat{m}_\gamma = m_{l_0}(q, p, u + 2\pi/\omega) - m_{l_0}(q, p, u) = \hat{2}$$

(intuitively, one has to rotate a two-oriented line twice to get it back to its initial position); choosing $\nu_0 = i/\hbar$, $\hbar = h/2\pi$, $V_{a,\alpha}$ is thus quantized if and only if

$$\frac{n}{2} + \frac{n}{h} ma^2 \omega\pi \in \mathbb{Z}$$

for all $n \in \mathbb{Z}$, which condition is equivalent to

$$a^2 = (2n+1)\hbar/m^2\omega, \quad n = 0, 1, 2, \ldots, \tag{3.7}$$

hence the quantization of the trajectories of the harmonic oscillator in phase space; reporting the values of a^2 given by (3.7) in the classical formula $E = \frac{1}{2}ma^2\omega^2$ giving the total energy of the oscillator (3.6), we get the usual energy levels

$$E_n = (n + \tfrac{1}{2})\hbar\omega, \quad n = 0, 1, 2, \ldots \tag{3.8}$$

predicted by Quantum Mechanics, and which are the eigenvalues E_n of the Schrödinger operator

$$H(q, p) = -\frac{\hbar^2}{2m}\frac{d^2}{dq^2} + \tfrac{1}{2}m\omega^2 q^2$$

associated to the Hamiltonian (3.6). We do not explain here the agreement of the values given by (3.8) with these eigenvalues; instead we relate our quantization condition (3.5) to the quantization condition given by Leray [4, ch. II, §3,6, p. 144].

Let V be a connected Lagrangian manifold (in Z), equipped with a two-orientation denoted $\check{z} \mapsto \check{l}(\check{z})$. Let $R \in \mathrm{Mp}$; we define RV as being the image of V by r, the projection of R onto Sp; clearly RV is a Lagrangian manifold, and using the identification of Mp with $\langle \mathrm{Sp} \times \mathbb{Z}_4 \rangle_{l_0}$ (theorem 2.7)), together with theorem 2.2, it is also clear that RV is two-oriented; we will, in this situation, call R a *two-frame of Z* (it is not exactly the definition Leray gives of a two-frame [4, ch. I, §3,3]; it is, however, equivalent). Let now ψ be the *Lagrangian phase* of V; it is the function $\psi: \check{V} \to \mathbb{R}$, uniquely determined up to an additive constant, such that $d\psi(\check{z}) = \frac{1}{2}\omega(z, dz)$; the *phase* of V relative to the two-frame R is then

$$\varphi_R(\check{z}) = \psi(\check{z}) + \tfrac{1}{2}\langle p, x\rangle, \quad (x, p) = Rz, \quad z \text{ the projection of } \check{z}. \tag{3.9}$$

Let $\check{X}^* \in \Lambda_4$ have projection $X^* = \{0\} \times \mathbb{R}^n$, ref. [4, ch. I, §3,3] defines

$$m_R(\check{z}) = m'(R^{-1}X^*, \check{l}(\check{z})), \tag{3.10}$$

where m' is the restriction of m to the set $\{(\check{l}, \check{l}') : l \cap l' = \{0\}\}$ (see corollary 12.11); thus m_R is not defined on V, but only on $\check{V} \backslash \check{\Sigma}_R$, where $\check{\Sigma}_R = \{\check{z} \in \check{V} : rl(z) \cap X^* = \{0\}\}$ is the apparent contour of V relative to the two-frame R.

Now, Leray's quantization condition (called "Maslov's quantum condition" by Leray) can be stated as follows:

3.5. For every two-frame R, the function

$$\check{V}\backslash\check{\Sigma}_R \ni \check{z} \mapsto -\tfrac{1}{4}m_R(\check{z}) + \frac{\nu_0}{2\pi i}\,\hat{\psi}_R(\check{z}) \in \mathbb{R}/4\mathbb{Z}$$

is defined modulo 1 on $V\backslash\Sigma_R$ (Σ_R the projection of $\check{\Sigma}_R$ on V).

Remark. If condition 3.5 holds for one two-frame R, it holds for all two-frames [4, def. 6.2].

Proposition 3.6. *The Bohr–Sommerfeld quantization condition (eq. 3.5) and the Leray–Maslov quantization condition 3.5 are equivalent; they thus define the same quantized manifolds.*

Proof. In view of (3.9) the Leray–Maslov quantization condition is equivalent to

3.6.

$$G_R(\check{z}) = -\tfrac{1}{4}m_R(\check{z}) + \frac{\nu_0}{2\pi i}\,\hat{\psi}(\check{z}), \quad \check{z}\in\check{V}\backslash\check{\Sigma}_R,$$

is defined modulo 1 on $V\backslash\Sigma_R$.

Let $\gamma\in\pi_1(V)$; $\gamma\check{z}$ and $\check{z}$ have the same projection z, hence in particular $\gamma\check{\Sigma}_R = \check{\Sigma}_R$. In view of formulas ($2_1$) and (3) in theorem (2.4) we have

$$m_R(\gamma\check{z}) - m_R(\check{z}) = m_{R\check{\ell}^*}(\check{z}) - m_{R\check{\ell}^*}(\gamma\check{z})$$

for $\check{z}\in\check{V}\backslash\check{\Sigma}_R$; by proposition 3.3 the r.h.s. of this equality depends neither on R, nor on $\check{z}$, hence $m_R(\gamma\check{z}) - m_R(\check{z}) = -\hat{m}_\gamma$; on the other hand it is obvious that

$$\psi(\gamma\check{z}) - \psi(\check{z}) = \frac{1}{2}\oint_\gamma \omega(z, \mathrm{d}z) = \oint_\gamma p\,\mathrm{d}q,$$

hence $\psi(\gamma\check{z}) - \psi(\check{z}) = C_\gamma$; it thus follows from condition 3.6 that

$$+\tfrac{1}{4}\hat{m}_\gamma + \frac{\nu_0}{2\pi i}\,\hat{C}_\gamma \in \mathbb{Z}_4,$$

hence Leray–Maslov quantization implies Bohr–Sommerfeld quantization. Suppose conversely

$$\tfrac{1}{4}(m_{\hat{l}_0}(\gamma\check{z}) - m_{\hat{l}_0}(\check{z})) + \frac{\nu_0}{2\pi i}\,(\hat{\varphi}(\gamma\check{z}) - \hat{\varphi}(\check{z})) \in \mathbb{Z}_4,$$

that is,

$$\tfrac{1}{4}(m_{\hat{l}_0}(\gamma\check{z}) - m_{\hat{l}_0}(\check{z})) + \frac{\nu_0}{2\pi i}\,(\hat{\psi}(\gamma\check{z}) - \psi(\check{z})) \in \mathbb{Z}_4$$

for all $\gamma \in \pi_1(V)$; choosing an element R in Mp such that $Rl_0 = X^*$, which is possible since Mp acts transitively on Λ_4, we can rewrite that relation as

$$\tfrac{1}{4}(m(\hat{l}(\gamma\check{z}), R^{-1}\hat{X}^*) - m(\hat{l}(\check{z}), R^{-1}\hat{X}^*)) + \frac{\nu_0}{2\pi \mathrm{i}}(\hat{\psi}(\gamma\check{z}) - \hat{\psi}(\check{z})) \in \mathbb{Z}_4 ,$$

which is equivalent to 3.6 for $\check{z} \in \check{V} \setminus \check{\Sigma}_R$ again in view of formula (2_1) in proposition 2.4.

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